Permanent Approach to Order Statistics and Robustness

Prof. N. Balakrishnan

Dept. of Mathematics & Statistics

McMaster University

Hamilton, Canada

bala@mcmaster.ca

Based on the Recent Article

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“Permanents, Order Statistics, Outliers, and Robustness”,

Revista Matemática Complutense, **20**, 7–107 (2007).

Roadmap

1. Order Statistics
2. Single-Outlier Model
3. Permanents
4. INID Model
5. Multiple-Outlier Model
6. Exponential Case
7. Robustness Issue
8. Logistic Case
9. Other Cases
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11. Ordered Ranked Set Sampling
12. Bibliography

Order Statistics

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- If we arrange these X_i's in increasing order of magnitude, we obtain the so-called *order statistics*, denoted by

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n},$$

which are clearly dependent.

Order Statistics (cont.)

- The pdf of $X_{r:n}$ is (for $x \in \boldsymbol{R}$)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r} f(x).$$

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- Similarly, the joint pdf of $(X_{r:n}, X_{s:n})$ as (for $1 \le r < s \le n$ and $x < y$)

$$\begin{aligned} f_{r,s:n}(x,y) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \{F(x)\}^{r-1} f(x) \\ &\quad \times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} f(y). \end{aligned}$$

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- The area of order statistics has a long and rich history, and a very vast literature.

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  - David (1970, 1981)
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- Among the many known results, the *triangle rule* is (for $1 \leq r \leq n-1$)

$$r f_{r+1:n}(x) + (n-r) f_{r:n}(x) = n f_{r:n-1}(x) \forall x \in \boldsymbol{R}.$$

Order Statistics (cont.)

- Similarly, the *rectangle rule* is ($2 \le r < s \le n$, $x < y$)

$$(r-1)f_{r,s:n}(x,y) + (s-r)f_{r-1,s:n}(x,y)$$
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- Among many more interesting results is the following.
- Let $X_1, X_2, \cdots, X_n$ be a random sample from a symmetric (about 0) population with pdf $f(x)$, cdf $F(x)$. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from the corresponding folded distribution with pdf and cdf

$$g(x) = 2f(x) \quad \text{and} \quad G(x) = 2F(x) - 1 \ \text{ for } x > 0.$$

Order Statistics (cont.)

Let $X_{r:n}$ and $Y_{r:n}$ be the corresponding order statistics, and $\left(\mu_{r:n}^{(k)}, \mu_{r,s:n}\right)$ and $\left(\nu_{r:n}^{(k)}, \nu_{r,s:n}\right)$ denote their single and product moments, respectively. We then have:

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$$\mu_{r:n}^{(k)} = \frac{1}{2^n}\left\{\sum_{i=0}^{r-1}\binom{n}{i}\nu_{r-i:n-i}^{(k)} + (-1)^k\sum_{i=r}^{n}\binom{n}{i}\nu_{i-r+1:i}^{(k)}\right\};$$

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  - *Trimmed Means*
  - *Winsorized Means*
  - *Linearly Weighted Means, etc.*

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- A *Single-Outlier Model (S-O Model)* simply stipulates that the sample contains IID observations $X_1, \cdots, X_{n-1}$ from a pdf $f(x)$ and one independent observation Y from another pdf $g(x)$.
- While $f(\cdot)$ and $g(\cdot)$ can be any two densities, it is common to assume that $g(x)$ corresponds to a scale and/or location shift of $f(x)$.

S-O Model (cont.)

- Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ be the order statistics obtained by arranging $(X_1, \cdots, X_{n-1}, Y)$ in increasing order of magnitude.

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$$(-\infty, x], \quad (x, x + \delta x] \quad \text{and} \quad (x + \delta x, \infty),$$

we obtain the pdf of $Z_{r:n}$ (for $r = 1, 2, \cdots, n$) as

S-O Model (cont.)

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\begin{aligned}
f_{r:n}(x) &= \frac{(n-1)!}{(r-2)!(n-r)!}\{F(x)\}^{r-2}G(x) \\
&\qquad \times f(x)\{1-F(x)\}^{n-r} \\
&+ \frac{(n-1)!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}g(x)\{1-F(x)\}^{n-r} \\
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& + \frac{(n-1)!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}g(x)\{1-F(x)\}^{n-r} \\
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where first and last terms vanish when $r = 1$ and n.

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\end{aligned}
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where first and last terms vanish when $r = 1$ and n.

- Similarly, the joint density of $(Z_{r:n}, Z_{s:n})$ will have five terms depending on which of the five intervals the outlier Y falls in.

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- For this reason, majority of the work in outlier literature deal with only *Single-Outlier Model*; see

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- We, therefore, need a different approach to handle multiple outliers.

Permanents

- Suppose $\boldsymbol{A} = ((a_{i,j}))$ is a square matrix of order n. Then, the *permanent* of the matrix $\boldsymbol{A}$ is defined to be

$$Per\,[\boldsymbol{A}] = \sum_{P} \prod_{i=1}^{n} a_{i,P(i)},$$

where $\sum_P$ denotes the sum over all $n!$ permutations $(P(1), P(2), \ldots, P(n))$ of $(1, 2, \ldots, n)$.

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- The above definition is similar to that of a determinant, except that it does not have the alternating sign.
- So, it is not surprising to see the following basic properties of permanents.

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- If $\boldsymbol{A}(i,j)$ denotes the sub-matrix of order $n-1$ obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column, then

$$Per\,[\boldsymbol{A}] = \sum_{i=1}^{n} a_{i,j}\; Per\,[\boldsymbol{A}(i,j)] = \sum_{j=1}^{n} a_{i,j}\; Per\,[\boldsymbol{A}(i,j)]\,;$$

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  i.e., permanent is expandable by any row (column).

- Due to the absence of the alternating sign, a permanent in which two or more rows (or columns) are repeated **need not be zero** (unlike a determinant).

Permanents (cont.)

- If $\boldsymbol{A}^*$ denotes the matrix obtained from $\boldsymbol{A}$ simply by replacing the ith row by $c\, a_{i,j}$ $(j = 1, \ldots, n)$, then

$$Per\,[\boldsymbol{A}^*] = c \;\; Per\,[\boldsymbol{A}]\,.$$

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- If $\boldsymbol{A}^{**}$ denotes the matrix obtained from $\boldsymbol{A}$ by replacing the ith row by $a_{i,j} + b_{i,j}\ (j = 1, \ldots, n)$ and $\boldsymbol{A}^*$ the matrix obtained from $\boldsymbol{A}$ by replacing the ith row by $b_{i,j}\ (j = 1, \ldots, n)$, then

$$Per\,[\boldsymbol{A}^{**}] = Per\,[\boldsymbol{A}] + Per\,[\boldsymbol{A}^*]\,.$$

Permanents (cont.)

- Let

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \dots & \dots & \dots & \dots \end{pmatrix}\begin{array}{l} \} \; i_1 \\ \} \; i_2 \\ \\ \end{array}$$

denote a matrix in which first row is repeated i_1 times, second row is repeated i_2 times, and so on.

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  denote a matrix in which first row is repeated i_1 times, second row is repeated i_2 times, and so on.

- We will now use the idea of permanents to study order statistics from n independent non-identically distributed (INID) variables $X_i \sim (F_i(x), f_i(x))\,,\ i = 1, \cdots, n$.

INID Model

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&\quad \times \prod_{\ell=r+1}^{n} \left\{1 - F_{P(\ell)}(x)\right\},
\end{aligned}
$$

where $(P(1), \cdots, P(r-1)), P(r), (P(r+1), \cdots, P(n))$ are mutually exclusive subsets of permutation $(P(1), \cdots, P(n))$ of $(1, \cdots, n)$.

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& \times f_{P(r)}(x) \prod_{\ell=r+1}^{s-1} \left\{ F_{P(\ell)}(y) - F_{P(\ell)}(x) \right\} \\
& \times f_{P(s)}(y) \prod_{\ell=s+1}^{n} \left\{ 1 - F_{P(\ell)}(y) \right\}, \ x < y,
\end{aligned}
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where $(P(1), \cdots, P(r-1)), P(r), (P(r+1), \cdots, P(s-1))$, $P(s)$, $(P(s+1), \cdots, P(n))$ are mutually exclusive subsets of permutation $(P(1), \cdots, P(n))$ of $(1, \cdots, n)$.

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for $r = 1, \cdots, n$ and $x \in \boldsymbol{R}$.

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for $1 \le r < s \le n$ and $x < y$.

INID Model (cont.)

- Triangle Rule: For $1 \leq r \leq n-1$ and $x \in \boldsymbol{R}$,

$$r\ f_{r+1:n}(x) + (n-r)\ f_{r:n}(x) = \sum_{i=1}^{n} f^{[i]}_{r:n-1}(x),$$

where $f^{[i]}_{r:n-1}(x)$ is the pdf of r^{th} order statistic among $X_1, \cdots, X_n$ with X_i removed.

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where $f_{r:n-1}^{[i]}(x)$ is the pdf of r^{th} order statistic among $X_1, \cdots, X_n$ with X_i removed.

Proof: For $1 \le r \le n-1$, we have

$$\begin{aligned} r\ f_{r+1:n}(x) &= \frac{1}{(r-1)!(n-r-1)!} \\ &\quad \times Per \begin{array}{c} \left[\begin{array}{ccc} F_1(x) & \cdots & F_n(x) \\ f_1(x) & \cdots & f_n(x) \\ 1-F_1(x) & \cdots & 1-F_n(x) \end{array}\right] \end{array} \begin{array}{l} \}\ r \\ \}\ 1 \\ \}\ n-r-1 \end{array} \end{aligned}$$

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$$(n-r)\ f_{r:n}(x) = \sum_{i=1}^{n} \{1 - F_i(x)\}\ f_{r:n-1}^{[i]}(x).$$

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$$(n-r)\ f_{r:n}(x) = \sum_{i=1}^{n} \{1 - F_i(x)\}\ f_{r:n-1}^{[i]}(x).$$

Adding the above two expressions, we get the result.

INID Model (cont.)

Proceeding similarly, we can establish the following result.

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- Rectangle Rule: For $2 \leq r < s \leq n$ and $x < y$,

$$
\begin{aligned}
&(r-1)\, f_{r,s:n}(x,y) + (s-r)\, f_{r-1,s:n}(x,y) \\
&\quad + (n-s+1)\, f_{r-1,s-1:n}(x,y) = \sum_{i=1}^{n} f^{[i]}_{r-1,s-1:n-1}(x,y),
\end{aligned}
$$

where $f^{[i]}_{r-1,s-1:n-1}(x,y)$ is the joint density of $\left(r^{\text{th}}, s^{\text{th}}\right)$ order statistics among $X_1, \cdots, X_n$ with X_i removed.

INID Model (cont.)

- Relations between two sets of OS: Let us consider $X_i \sim (F_i(x), f_i(x))\,,\ i = 1, \cdots, n$, as independent random variables, and $X_{1:n} \leq \cdots \leq X_{n:n}$ as the corresponding order statistics.

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- Relations between two sets of OS: Let us consider $X_i \sim (F_i(x), f_i(x))$, $i = 1, \cdots, n$, as independent random variables, and $X_{1:n} \leq \cdots \leq X_{n:n}$ as the corresponding order statistics.

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  Let $f_i(x)$ be all symmetric about 0.

  Let $Y_i \sim (G_i(x), g_i(x))$, $i = 1, \cdots, n$, be the corresponding folded (about 0) variables with

  $$g_i(x) = 2\, f_i(x) \text{ and } G_i(x) = 2F_i(x) - 1 \text{ for } x > 0,$$

  and $Y_{1:n} \leq \cdots \leq Y_{n:n}$ be the corresponding order statistics.

INID Model (cont.)

Let $\left(\mu_{r:n}^{(k)}, \mu_{r,s:n}\right)$ and $\left(\nu_{r:n}^{(k)}, \nu_{r,s:n}\right)$ denote the moments of OS $(X_{1:n} \leq \cdots \leq X_{n:n})$ and $(Y_{1:n} \leq \cdots \leq Y_{n:n})$.

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- Then, for $r = 1, \cdots, n$ and $k \geq 0$,

$$\mu_{r:n}^{(k)} = \frac{1}{2^n}\left\{\sum_{\ell=0}^{r-1} \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \nu_{r-\ell:n-\ell}^{(k)[i_1,\cdots,i_\ell]} + (-1)^k \sum_{\ell=r}^{n} \sum_{1 \leq i_1 < \cdots < i_{n-\ell} \leq n} \nu_{\ell-r+1:\ell}^{(k)[i_1,\cdots,i_{n-\ell}]}\right\},$$

where $\nu_{r:n-\ell}^{(k)[i_1,\cdots,i_\ell]}$ is the k^{th} moment of the r^{th} OS from $Y_1, \cdots, Y_n$ with $Y_{i_1}, \cdots, Y_{i_\ell}$ removed.

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&\quad+\sum_{\ell=s}^{n}\sum_{1\leq i_1<\cdots<i_{n-\ell}\leq n}\nu_{\ell-s+1,\ell-r+1:\ell}^{[i_1,\cdots,i_{n-\ell}]}\\
&\quad\left.-\sum_{\ell=r}^{s-1}\sum_{1\leq i_1<\cdots<i_\ell\leq n}\nu_{s-\ell:n-\ell}^{[i_1,\cdots,i_\ell]}\,\nu_{\ell-r+1:\ell}^{[i_{\ell+1},\cdots,i_n]}\right\},
\end{aligned}$$

where $\nu_{r,s:n-\ell}^{[i_1,\cdots,i_\ell]}$ is the product moment of the $\left(r^{\text{th}}, s^{\text{th}}\right)$ OS from $Y_1,\cdots,Y_n$ with $Y_{i_1},\cdots,Y_{i_\ell}$ removed.

Multiple-Outlier Model

- Now, let us consider the p-outlier model

$$F_1 = \cdots = F_{n-p} \equiv F(x) \text{ and } F_{n-p+1} = \cdots = F_n \equiv G(x).$$

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- Then, the generalized results of the type presented could be used to carry out exact computations efficiently for *multiple-outlier model (M-O Model)*.
- For example, the *triangle rule* becomes

$$r\ \mu_{r+1:n}^{(k)} + (n-r)\ \mu_{r:n}^{(k)} = (n-p)\ \mu_{r:n-1}^{(k)}[p] + p\ \mu_{r:n-1}^{(k)}[p-1],$$

where $\mu_{r:n-1}^{(k)}[p]$ and $\mu_{r:n-1}^{(k)}[p-1]$ are the moments when there are p and $p-1$ outliers, respectively.

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M-O Model (cont.)

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  "A study of the multiple-outlier model has been recently carried out by Balakrishnan, who gives a substantial body of results on the moments of order statistics. He indicated that these results can in principle be applied to robustness studies in the multiple-outlier situation, but at the time of writing, we are not aware of any published application. There is much work waiting to be done in this important area."

Exponential Case

- Consider the case when the variables X_i $(i = 1, \cdots, n)$ are independent with

$$f_i(x) = \frac{1}{\theta_i}\, e^{-x/\theta_i} \text{ and } F_i(x) = 1 - e^{-x/\theta_i}, \; x \geq 0, \; \theta_i > 0.$$

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- Then, these differential equations can be used along with the permanents approach to establish the following results for moments of order statistics.

Exponential Case (cont.)

- Result 1: For $n = 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$,

$$\mu_{1:n}^{(k+1)} = \frac{k+1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} \, \mu_{1:n}^{(k)}.$$

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- Result 2: For $2 \le r \le n$ and $k = 0, 1, 2, \cdots$,

$$\mu_{r:n}^{(k+1)} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} \left\{ (k+1)\mu_{r:n}^{(k)} + \sum_{i=1}^{n} \frac{1}{\theta_i} \, \mu_{r-1:n-1}^{(k+1)[i]} \right\}.$$

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- Result 3: For $n = 2, 3, \cdots$,

$$\mu_{1,2:n} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} \left\{ \mu_{1:n} + \mu_{2:n} \right\}.$$

Exponential Case (cont.)

- Result 4: For $2 \leq r \leq n-1$,

$$\mu_{r,r+1:n} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} \left\{ \mu_{r:n} + \mu_{r+1:n} + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1,r:n-1}^{[i]} \right\}.$$

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- Result 6: For $2 \le r < s \le n$ and $s - r \ge 2$,

$$\mu_{r,s:n} = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} \left\{ \mu_{r:n} + \mu_{s:n} + \sum_{i=1}^{n} \frac{1}{\theta_i} \mu_{r-1,s-1:n-1}^{[i]} \right\}.$$

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- Then, Results 1 and 2, for example, reduce to

$$
\begin{aligned}
\mu_{1:n}^{(k+1)}[p] &= \frac{k+1}{\frac{n-p}{\theta} + \frac{p}{\tau}}\, \mu_{1:n}^{(k)}[p]; \\
\mu_{r:n}^{(k+1)}[p] &= \frac{1}{\frac{n-p}{\theta} + \frac{p}{\tau}} \left\{ (k+1)\mu_{r:n}^{(k)}[p] + \frac{n-p}{\theta}\mu_{r-1:n-1}^{(k+1)}[p] \right. \\
&\qquad \left. + \frac{p}{\tau}\mu_{r-1:n-1}^{(k+1)}[p-1] \right\}.
\end{aligned}
$$

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  first two single and product moments of OS from a single-outlier model can be produced.

- These can be used to produce single and product moments of OS from a two-outlier model, and so on.

Robustness Issue

Optimal Winsorized estimator of θ and relative efficiency when $h = \frac{\theta}{\tau}$ and $n = 15$ [a]

	p=1		p=2		p=3		p=4	
h	m^*	RE	m^*	RE	m^*	RE	m^*	RE
0.50	15	1.000	14	1.048	13	1.104	12	1.161
0.40	14	1.084	13	1.237	12	1.404	10	1.555
0.30	14	1.329	12	1.793	10	2.222	9	2.543
0.20	13	2.222	11	3.628	9	4.777	7	5.583
0.10	13	7.649	10	14.355	8	19.249	6	22.423

[a] Winsorized mean $W_{m,n} = \frac{1}{m+1}\left\{\sum_{i=1}^{m-1} X_{i:n} + (n-m+1)X_{m:n}\right\}$.

Robustness Issue (cont.)

Optimal Trimmed estimator of θ and relative efficiency when $h = \frac{\theta}{\tau}$ and $n = 15$ [a]

	p=1		p=2		p=3		p=4	
h	m^*	RE	m^*	RE	m^*	RE	m^*	RE
0.50	14	0.982	14	1.185	14	1.378	13	1.537
0.40	14	1.051	14	1.313	13	1.511	13	2.000
0.30	14	1.140	14	1.350	13	1.864	13	2.217
0.20	14	1.229	13	1.558	13	1.996	12	2.776
0.10	14	1.314	13	1.838	12	2.457	11	3.128

[a] *Trimmed mean* $T_{m,n} = \frac{1}{m} \sum_{i=1}^{m} X_{i:n}$.

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- Once p is determined, we find $W_{n-p,n}$ as a provisional estimate of θ (say, $\tilde{\theta}$), then estimate h from the equation

$$nW_{n,n} = \left(n - p + \frac{p}{h}\right)\tilde{\theta},$$

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  and then determine m^* from the tables.
- Next, the corresponding $W_{m^*,n}$ may be used in place of $\tilde{\theta}$ in the above equation, and a new m^* be determined.
- Continue until m^* is stable, and use $W_{m^*,n}$ as estimate.

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Bias of Winsorized and Trimmed estimators of θ and relative efficiency when $h = \frac{\theta}{\tau} = 0.10$ and $n = 20$

$Estimator$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$W_{20,20}$	0.3810	0.8095	1.2381	1.6667
$W_{18,20}$	0.0528	0.2029	0.5246	0.9360
$T_{18,20}$	-0.1594	-0.0615	0.1103	0.3453
$W_{16,20}$	0.0241	0.1261	0.2568	0.4360
$T_{16,20}$	-0.3307	-0.2737	-0.2038	-0.1144

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- When p increases, Winsorized mean develops serious bias, but not Trimmed mean.

Robustness Issue (cont.)

Bias and MSE of estimators of θ when p outliers are present in the sample with $h = \frac{\theta}{\tau}$ and $n = 20$ [a]

		p=1		p=2		p=3	
h	Est	$Bias$	MSE	$Bias$	MSE	$Bias$	MSE
1.00	$W_{n,n}$	-0.048	0.048				
	$W_{.9n,n}$	-0.053	0.053				
	$T_{.9n,n}$	-0.233	0.088				
	CK_n	-0.073	0.048				
0.25	$W_{n,n}$	0.095	0.088	0.238	0.170	0.381	0.293
	$W_{.9n,n}$	0.020	0.060	0.107	0.084	0.213	0.141
	$T_{.9n,n}$	-0.181	0.071	-0.119	0.060	-0.047	0.057
	CK_n	0.065	0.078	0.202	0.146	0.339	0.252

[a] CK_n is Chikkagoudar-Kunchur estimator of θ

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- Trimmed estimator performs quite efficiently, and the gain in efficiency is substantial as compared to all other estimators.
- It is important to note that the greater protection provided by trimmed estimator (to the presence of one or more extreme outliers) comes at a higher premium.

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> *"When confronted with Professor Balakrishnan's results with myriad relations among moments of non-homogeneous exponential order statistics, lack of memory property could be used to produce alternate formulas. But, there would be little gain in efficiency when compared to Bala's algorithm. Bala's specialized differential equation techniques may perhaps have their finest hour in dealing with logistic case for which minima and maxima are not nice. His proposed work in this direction will be interesting."*

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$$f_i(x) = \frac{\frac{\pi}{\sqrt{3}} \exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}}{\sigma_i\left(1+\exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}\right)^2}, \quad x \in \boldsymbol{R},$$

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$$f_i(x) = \frac{\frac{\pi}{\sqrt{3}} \exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}}{\sigma_i\left(1+\exp\left\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu_i}{\sigma_i}\right)\right\}\right)^2}, \quad x \in \boldsymbol{R},$$

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$$f_i(x) = \frac{\pi}{\sigma_i\sqrt{3}} F_i(x)\left\{1 - F_i(x)\right\}, \; x \in \boldsymbol{R}.$$

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- Then, the differential equations can be used along with the permanents approach to establish the following results for moments of order statistics.

Logistic Case (cont.)

- Result 1: For $n = 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$,

$$\sum_{i=1}^{n} \frac{1}{\sigma_i} \, \mu_{1:n+1}^{(k+1)[i]^+} = -\frac{(k+1)\sqrt{3}}{\pi} \, \mu_{1:n}^{(k)} + \left(\sum_{i=1}^{n} \frac{1}{\sigma_i} \right) \mu_{1:n}^{(k+1)}.$$

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- Result 2: For $2 \le r \le n$ and $k = 0, 1, 2, \cdots$,

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{\sigma_i}\, \mu_{r:n+1}^{(k+1)[i]^+} \quad &= \quad \frac{(k+1)\sqrt{3}}{\pi} \left\{ \mu_{r-1:n}^{(k)} - \mu_{r:n}^{(k)} \right\} - \sum_{i=1}^{n} \frac{1}{\sigma_i}\, \mu_{r-1:n-1}^{(k+1)[i]} \\ &\quad + \left(\sum_{i=1}^{n} \frac{1}{\sigma_i}\right) \left\{ \mu_{r-1:n}^{(k+1)} + \mu_{r:n}^{(k+1)} \right\}. \end{aligned}$$

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- Result 3: For $n = 2, 3, \cdots$ and $k = 0, 1, 2, \cdots$,

$$\sum_{i=1}^{n} \frac{1}{\sigma_i}\, \mu_{n+1:n+1}^{(k+1)[i]^+} = \frac{(k+1)\sqrt{3}}{\pi}\, \mu_{n:n}^{(k)} + \left(\sum_{i=1}^{n} \frac{1}{\sigma_i}\right) \mu_{n:n}^{(k+1)}.$$

Logistic Case (cont.)

- In the case of p-outlier model given by

$$(X_1, \cdots, X_{n-p}) \sim L(\mu, \sigma) \text{ and } (X_{n-p+1}, \cdots, X_n) \sim L(\mu_1, \sigma_1),$$

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  these reduce to the following results:

- For $n = 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$,

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\mu_{1:n+1}^{(k+1)}[p+1] &= \frac{\sigma_1}{p}\left\{\left(\frac{n-p}{\sigma} + \frac{p}{\sigma_1}\right)\mu_{1:n}^{(k+1)}[p] - \frac{n-p}{\sigma}\mu_{1:n+1}^{(k+1)}[p]\right. \\
&\quad \left. - \frac{(k+1)\sqrt{3}}{\pi}\mu_{1:n}^{(k)}[p]\right\}; \\
\mu_{n+1:n+1}^{(k+1)}[p+1] &= \frac{\sigma_1}{p}\left\{\left(\frac{n-p}{\sigma} + \frac{p}{\sigma_1}\right)\mu_{n:n}^{(k+1)}[p] - \frac{n-p}{\sigma}\mu_{n+1:n+1}^{(k+1)}[p]\right. \\
&\quad \left. + \frac{(k+1)\sqrt{3}}{\pi}\mu_{n:n}^{(k)}[p]\right\}; \\
\cdots & \quad \cdots
\end{aligned}$$

Logistic Case (cont.)

Bias of estimators of the mean of a logistic distribution

when $p = 1$ outlier is present in the sample with

$\mu_0 = 0$, $\sigma = \sigma_1 = 1$ and $n = 20$

	μ_1				
$Estimator$	0.5	1.0	2.0	3.0	4.0
$Mean$	0.0250	0.0500	0.1000	0.1500	0.2000
$Trim(10\%)$	0.0245	0.0459	0.0728	0.0817	0.0836
$Trim(20\%)$	0.0241	0.0434	0.0626	0.0672	0.0681
$Wins(10\%)$	0.0248	0.0479	0.0812	0.0943	0.0974
$Wins(20\%)$	0.0244	0.0451	0.0683	0.0745	0.0756
$LWMean(10\%)$	0.0240	0.0432	0.0624	0.0673	0.0682
$LWMean(20\%)$	0.0239	0.0420	0.0585	0.0620	0.0627
$Median$	0.0236	0.0407	0.0548	0.0576	0.0581

Logistic Case (cont.)

Bias of estimators of the mean of a logistic distribution

when $p = 2$ outliers are present in the sample with

$\mu_0 = 0$, $\sigma = \sigma_1 = 1$ and $n = 20$

	μ_1				
$Estimator$	0.5	1.0	2.0	3.0	4.0
$Mean$	0.500	0.1000	0.2000	0.3000	0.4000
$Trim(10\%)$	0.0491	0.0933	0.1562	0.1862	0.1968
$Trim(20\%)$	0.0485	0.0887	0.1332	0.1458	0.1482
$Wins(10\%)$	0.0496	0.0969	0.1751	0.2224	0.2420
$Wins(20\%)$	0.0490	0.0920	0.1464	0.1643	0.1680
$LWMean(10\%)$	0.0484	0.0883	0.1328	0.1467	0.1500
$LWMean(20\%)$	0.0480	0.0861	0.1236	0.1327	0.1343
$Median$	0.0476	0.0836	0.1153	0.1219	0.1231

Other Cases

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  - *Normal distribution*
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  - *Pareto distribution*
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- Let $X_1, \cdots, X_n$ be independent variables with continuous distributions $F_1, \cdots, F_n$ and densities $f_1, \cdots, f_n$, respectively.

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- Let $X_{1:m:n}^{(R_1,\cdots,R_m)} \leq \cdots \leq X_{m:m:n}^{(R_1,\cdots,R_m)}$ be the progressively Type-II censored order statistics obtained by using the progressive censoring scheme $(R_1, \cdots, R_m)$.

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- Balakrishnan and Cramer (2007) derived the joint density of progressively Type-II censored sample $(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)})$ as

Progressive Censoring (cont.)

$$f_{X_{1:m:n},\cdots,X_{m:m:n}}(x_1,\cdots,x_m)$$

$$= \frac{1}{(n-1)!}\left(\prod_{j=2}^{m}\gamma_j\right)$$

$$\times \text{ Per} \begin{array}{c} \left(\begin{array}{ccc} f_1(x_1) & \cdots & f_n(x_1) \\ 1-F_1(x_1) & \cdots & 1-F_n(x_1) \\ \cdot & \cdots & \cdot \\ f_1(x_m) & \cdots & f_n(x_m) \\ 1-F_1(x_m) & \cdots & 1-F_n(x_m) \end{array}\right) \end{array} \begin{array}{l} \}1 \\ \}R_1 \\ \cdot \\ \}1 \\ \}R_m \end{array} ,$$

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where $\gamma_j = \sum_{i=j}^{m}(R_i+1),\ j=1,\cdots,m.$

Progressive Censoring (cont.)

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Progressive Censoring (cont.)

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- They also used these results to examine the effect of an outlier in the underlying sample and the robust estimation of the exponential mean when an outlier is possibly present in the progressively Type-II censored sample.

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$$X_{1:n}^{\text{ORSS}} \leq \cdots \leq X_{n:n}^{\text{ORSS}}.$$

Ordered RSS (cont.)

- The joint density of $X_{1:n}^{\text{ORSS}} \leq \cdots \leq X_{n:n}^{\text{ORSS}}$ is (for $x_1 < x_2 < \cdots < x_n$)

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$$= \text{Per} \begin{pmatrix} f_{1:n}(x_1) & \cdots & f_{n:n}(x_1) \\ f_{1:n}(x_2) & \cdots & f_{n:n}(x_2) \\ \cdot & \cdots & \cdot \\ f_{1:n}(x_{n-1}) & \cdots & f_{n:n}(x_{n-1}) \\ f_{1:n}(x_n) & \cdots & f_{n:n}(x_n) \end{pmatrix} \begin{array}{l} \}1 \\ \}1 \\ \cdot \\ \}1 \\ \}1 \end{array}$$

Ordered RSS (cont.)

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- Balakrishnan and Li (2006, 2007) used it to develop optimal nonparametric and parametric inference based on ORSS.

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