Heisenberg uniqueness pairs and positive matrices

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Abstract

A Heisenberg uniqueness pair (HUP) is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any bounded Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length, and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure. For instance, when Γ is the hyperbola $x_1 x_2 = 1$, and Λ is the lattice-cross

$$(\alpha\mathbf{Z} \times \{0\}) \cup (0 \times \mathbf{Z}),$$

where α and β are positive reals, then (Γ, Λ) is an HUP if and only if $\alpha\beta \leq 1$; in this situation, the Fourier transform $\widehat{\mu}$ of the measure solves the one-dimensional Klein-Gordon equation. Phrased differently, this particular problem is equivalent to the fact that

$$e^{i\pi n t}, e^{i\pi n/t}, \quad n \in \mathbf{Z},$$

span a weak-star dense subspace in $L^1(\mathbf{R})$ if and only if $\alpha\beta \leq 1$. In order to prove this kind of theorems, some elements of linear fractional theory and ergodic theory are needed, such as the Birkhoff Ergodic Theorem. In this connection a number of questions related with positive matrices will be apparent.

Keywords

Heisenberg uniqueness pair, Perron-Frobenius operator, Positive matrices, Klein-Gordon equation.