Existence and representation of the group inverse of additive perturbations

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Abstract

Let R be a ring with unity 1. An element $a \in R$ is said to be von Neumann regular if there exists an element a^- of R such that $aa^-a = a$. In this case, a^- is called a $\{1\}$-inverse of a. An element a^+ of R is a $\{1,2\}$-inverse of a if $aa^+a = a$ and $a^+aa^+ = a^+$ hold. An element $a^\sharp \in R$ is said to be the group inverse of a if $a^\sharp$ is a $\{1,2\}$-inverse of a such that $aa^\sharp = a^\sharp a$. If a is a unit of R, then $a^\sharp = a^{-1}$. A von Neumann regular element $a \in R$ is group invertible if and only if $a + 1 - aa^-$ is a unit of R, independently of the choice of a^- of a. [2, Theorem 3.1]. The existence of the group inverse of a was characterized in [4] by means of other unit, $1 - aa^- - a^-a$. If the group inverse of a exists, the additive perturbed element $a+b$ is not necessarily group invertible. Our purpose is to give a characterization for the group invertibility of $a+b$ in terms of a unit of R. Further, a representation for $(a+b)^\sharp$ will be derived. The group inverse plays an important role in the theory of Markov chains [1]. Some applications in the setting of complex square matrices will be indicated.

The research is partially supported by Project MTM2010-18057, "Ministerio de Ciencia e Innovación" of Spain.

Keywords

Von Neumann regular, Group inverse, $\{1,2\}$-inverse.

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